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1992 J. Phys. A: Math. Gen. 25 L183

(http://iopscience.iop.org/0305-4470/25/4/006)

J. Phys. A: Math. Gen. 25 (1992) L183–L187. Printed in the UK

LETTER TO THE EDITOR

Mott scattering of anyons: some exact results

J Law†§, M K Srivastava†||, R K Bhaduri† and Avinash Khare‡

† Physics Department, McMaster University, Hamilton, Ontario, Canada L8S 4M1¶
‡ Institute of Physics, Bhubaneswar 751005, India

Received 14 October 1991

Abstract. The planar differential cross section for the Mott scattering of two identical charged anyons is obtained in a closed form. This expression exhibits a marked asymmetry between forward and backward angles, and is especially simple for the semion–semion scattering. At low energies, a pronounced dip in the differential cross section due to planar Coulomb–anyonic interference appears at the forward angles, whose implication is discussed.

In this letter, the planar Coulomb scattering cross section of two anyons is obtained analytically, and the effect of the interference between the planar Coulomb and the statistical interaction on the cross section is examined. Here, by planar Coulomb, we mean a $1/r$ potential rather than a logarithmic potential, although it is the latter that obeys the Laplace equation in two dimensions. We thus have in mind a three-dimensional system in which the motion of the charged particles is confined in a plane, as in the quantum Hall effect [1]. The expression for the pure anyonic cross section was already contained in the classic paper by Aharonov and Bohm [2], and interest in this problem was revived recently by March-Russell and Wilczek [3]. Since then, others have studied anyonic scattering [4] or the related Dirac scattering of low-energy fermions by a Chern–Simons vortex [5], but no one to our knowledge has calculated the Mott scattering cross section of anyons. A number of studies have been made [6–9] for the three-dimensional Coulomb problem in the presence of the Bohm–Aharonov potential, and the Green function has been written as an infinite double sum. From the poles of the Green function, the bound states for the attractive case have also been deduced. But the non-trivial problem of obtaining the planar Coulomb cross section of anyons in a closed form and the explicit demonstration of the interference effects due to statistics of identical particles in this case are obtained here for the first time. Besides being an example of an exactly solvable non-trivial problem, the final expression for semions is remarkably simple. The Mott differential cross section for anyons exhibits a pronounced dip at forward angles, whose implication is discussed.

Consider the scattering of two identical anyons in the presence of a repulsive planar Coulomb interaction, $V(r)=e^2/r$, where $r=|\boldsymbol{r}|$, is the relative distance between the anyons in a plane. The Lagrangian for the relative motion in polar coordinates (r,θ) is given by

$$L=\tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2-\hbar\alpha\dot{\theta}-V(r) \qquad (1)$$

§ Permanent address: Department of Physics, University of Guelph, Guelph, Ontario, Canada.
|| Permanent address: Department of Physics, University of Roorkee, Roorkee 247667, India.
¶ Mailing address.

0305-4470/92/040183+05$04.50

where $\mu = m/2$ is the reduced mass, $\boldsymbol{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)$, and $\theta = \theta_{12} = \tan^{-1}[(y_1 - y_2)/(x_1 - x_2)]$. The corresponding Hamiltonian is

$$H = \frac{p_r^2}{2\mu} + \frac{(p_\theta + \hbar\alpha)^2}{2\mu r^2} + V(r) \tag{2}$$

with $p_r = (\partial L/\partial \dot{r})$, $p_\theta = (\partial L/\partial \dot{\theta})$. For $\alpha = 0$, the two particles are bosons interacting only via the planar Coulomb potential $V(r)$, while for $\alpha = 1$, they are fermions. We restrict the strength of the statistical interaction α to be in the range (0, 1), and the special case $\alpha = \frac{1}{2}$ corresponds to semions. The Schrödinger equation for each partial wave $\psi_l(r)$ may be written down, whose solutions are confluent hypergeometric functions, as for the Coulomb problem [10]. The regular solution is expressed as a linear combination of the partial waves, with appropriate coefficients c_l's, and is given by

$$\psi_k(r) = \sum_{l=-\infty}^{\infty} c_l\, \mathrm{e}^{\mathrm{i}l\theta} r^{|l+\alpha|}\, \mathrm{e}^{\mathrm{i}kr} M\left(|l+\alpha| + \frac{1}{2} + \frac{\mathrm{i}\beta}{2k}, 2|l+\alpha| + 1, -2\mathrm{i}kr\right). \tag{3}$$

Here $M(a, b, z)$ is the confluent hypergeometric function defined in Abramowitz and Stegun [11], and $\beta = me^2/\hbar^2$ is the inverse Bohr radius. The coefficients c_l are chosen such that when the planar Coulomb distorted asymptotic plane wave is subtracted from the solution (3), only an outgoing phase shifted spherical wave is left, modulated by the scattering amplitude $f_k(\theta)$, with θ the scattering angle. We obtain

$$f_k(\theta) = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{2\pi k \mathrm{i}}}\, \mathrm{e}^{\mathrm{i}l\theta} [\mathrm{e}^{2\mathrm{i}\eta_{|l+\alpha|} + \mathrm{i}\pi(|l| - |l+\alpha|)} - 1] \tag{4}$$

where

$$\mathrm{e}^{2\mathrm{i}\eta_{|l+\alpha|}} = \Gamma\left(|l+\alpha| + \frac{1}{2} + \frac{\mathrm{i}\beta}{2k}\right) \Big/ \Gamma\left(|l+\alpha| + \frac{1}{2} - \frac{\mathrm{i}\beta}{2k}\right).$$

In deriving the above equation, we have assumed the incident wave to be incident from the left. To recover the convention used by Aharonov and Bohm [2], our scattering angle θ should be replaced by $(\theta - \pi)$. Note also that (4) reduces to the correct well known limits for anyonic scattering for $\beta = 0$. It also gives the correct two-dimensional scattering cross section for a $1/r$ potential when $\alpha = 0$.

The series on the right-hand side of (4) may be expressed in terms of hypergeometric functions [11] by splitting the sum into ranges from $l = 0$ to ∞ and from $l = -1$ to $-\infty$. This may be done by using the relation

$$\mathrm{e}^{2\mathrm{i}\eta_\alpha}\, {}_2F_1\left(1, \frac{1}{2} + \alpha + \frac{\mathrm{i}\beta}{2k}; \frac{1}{2} + \alpha - \frac{\mathrm{i}\beta}{2k}; \mathrm{e}^{\mathrm{i}\theta}\right) = \sum_{l=0}^{\infty} \mathrm{e}^{2\mathrm{i}\eta_{l+\alpha}}\, \mathrm{e}^{\mathrm{i}l\theta}$$

and a similar one for the sum from $l = -1$ to $-\infty$.

The result is

$$f_k(\theta) = \frac{1}{\sqrt{2\pi k \mathrm{i}}} \left[\mathrm{e}^{2\mathrm{i}\eta_\alpha - \mathrm{i}\pi\alpha}\, {}_2F_1\left(1, \frac{1}{2} + \alpha + \frac{\mathrm{i}\beta}{2k}; \frac{1}{2} + \alpha - \frac{\mathrm{i}\beta}{2k}; \mathrm{e}^{\mathrm{i}\theta}\right) \right.$$
$$\left. + \mathrm{e}^{-\mathrm{i}\theta + 2\mathrm{i}\eta_{1-\alpha} + \mathrm{i}\pi\alpha}\, {}_2F_1\left(1, \frac{3}{2} - \alpha + \frac{\mathrm{i}\beta}{2k}; \frac{3}{2} - \alpha - \frac{\mathrm{i}\beta}{2k}; \mathrm{e}^{-\mathrm{i}\theta}\right) \right]. \tag{5}$$

We would like to point out that the hypergeometric functions in (5) diverge at $\theta=0$. For scattering in the pure Bohm–Aharonov case, the question of a spurious delta-function in the forward scattering amplitude due to the interchange of the l sum in the partial wave expansion with the $r\to\infty$ limit in the wavefunction has been examined carefully by Hagen [4]. It is more convenient to rewrite (5) in a form that apparently separates the planar Coulomb and anyonic parts and simplifies considerably for the special cases of $\alpha=0, \frac{1}{2}$ and 1. Some non-trivial algebra results in yielding the following form (with the restriction $|\arg(-z)|<\pi$, where $z=\mathrm{e}^{\mathrm{i}\theta}$, and excluding $\theta=0$) for $f_k(\theta)$:

$$f_k(\theta)=\frac{\exp\left\{\left(2\mathrm{i}\eta_\alpha-\mathrm{i}\pi\alpha-\frac{\mathrm{i}\beta}{2k}\ln\left(\sin^2\frac{\theta}{2}\right)\right]\right\}}{\sqrt{2\mathrm{i}k\sin^2\frac{\theta}{2}}}$$

$$\times\left[(-\mathrm{e}^{\mathrm{i}\theta})^{-\alpha}\frac{\left(\frac{-\mathrm{i}\beta}{2k}\right)\Gamma\left(\frac{1}{2}+\alpha-\frac{\mathrm{i}\beta}{2k}\right)\Gamma\left(\frac{1}{2}-\alpha-\frac{\mathrm{i}\beta}{2k}\right)}{\Gamma\left(1-\frac{\mathrm{i}\beta}{2k}\right)\Gamma\left(\frac{1}{2}-\frac{\mathrm{i}\beta}{2k}\right)}\right.$$

$$+\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{\pi}}(2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\theta})^{-(1+(\mathrm{i}\beta/k))}\left(\frac{\frac{1}{2}-\alpha+\frac{\mathrm{i}\beta}{2k}}{\frac{1}{2}-\alpha-\frac{\mathrm{i}\beta}{2k}}\right)$$

$$\times\left(\mathrm{e}^{2\pi\mathrm{i}\alpha}\frac{\cos\left[\pi\left(\alpha+\frac{\mathrm{i}\beta}{2k}\right)\right]}{\cos\left[\pi\left(\alpha-\frac{\mathrm{i}\beta}{2k}\right)\right]}-1\right)$$

$$\left.\times{}_2F_1\left(\frac{1}{2}-\alpha-\frac{\mathrm{i}\beta}{2k},-\frac{\mathrm{i}\beta}{k}1;\frac{3}{2}-\alpha-\frac{\mathrm{i}\beta}{2k};\mathrm{e}^{-\mathrm{i}\theta}\right)\right]. \qquad (6)$$

In the expression above, the second term in the square bracket on the right-hand side vanishes for $\alpha=0$, 1 and $\frac{1}{2}$, as may be easily checked. Furthermore, for $\alpha=1$, the factor $(-\mathrm{e}^{\mathrm{i}\theta})^{-\alpha}$ in the first term ensures that an overall negative sign appears in the scattering amplitude f_k as $\theta\to(\theta-\pi)$, appropriate for fermions. To elaborate on this point, note that in the bosonic basis, only even l's in the sums (3) and (4) are allowed. This may be taken care of in the usual fashion by summing over all integral l's but including the exchange amplitude $f_k(\theta-\pi)$. The expression for the differential cross section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta}=|f_k(\theta)+f_k(\theta-\pi)|^2. \qquad (7)$$

It is now relatively simple to calculate the differential cross section for any α in the range (0, 1), and the expressions are especially simple for the scattering cases: bosonic

($\alpha = 0$), fermionic ($\alpha = 1$) and semionic ($\alpha = \frac{1}{2}$). For the bosonic and fermionic scattering,

$$\frac{d\sigma}{d\theta} = \frac{\beta}{4k^2} \tanh\left(\frac{\pi\beta}{2k}\right) \left[\frac{1}{\sin^2 \frac{\theta}{2}} + \frac{1}{\cos^2 \frac{\theta}{2}} \pm \frac{2 \cos\left[\frac{\beta}{2k} \ln\left(\tan^2 \frac{\theta}{2}\right)\right]}{\sqrt{\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}}} \right] \tag{8}$$

where the plus(+) sign is for bosons and the minus(−) sign for fermions. In the case of semion–semion scattering, the differential cross section is given by

$$\frac{d\sigma}{d\theta} = \frac{\beta}{4k^2} \coth\left(\frac{\pi\beta}{2k}\right) \left[\frac{1}{\sin^2 \frac{\theta}{2}} + \frac{1}{\cos^2 \frac{\theta}{2}} + \frac{2 \sin\left[\frac{\beta}{2k} \ln\left(\tan^2 \frac{\theta}{2}\right)\right]}{\sqrt{\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}}} \right]. \tag{9}$$

It may also be easily seen that we recover the pure anyon–anyon scattering cross section, if we set $\beta = 0$ in (6) and substitute in (7).

We can make some general remarks about the scattering cross section as exhibited by the expressions above.

(*a*) An examination of (4) for $f_k(\theta)$ shows that it is invariant for $\theta \to -\theta$ and $\alpha \to -\alpha$. This corresponds to combined P and T transformations [3].

(*b*) Again from (4), it is clear that $f_k(\theta - \pi) = f_k(\theta + \pi)$. It then follows from (7) that the differential cross section is invariant under the rotation $\theta \to \theta + \pi$, i.e. $x \to -x$, and $y \to -y$.

The differential cross sections for scattering, multiplied by β, is dimensionless and depends only on the angle θ, and the parameter β/k for a given α. This quantity is plotted on a logarithmic scale in figure 1 as a function of the angle θ for $\alpha = 0$ (bosonic), $\alpha = 1$ (fermionic) and various intermediate values of α, including the special case of $\alpha = \frac{1}{2}$ (semionic). In this figure two sets of curves are shown. The upper set corresponds to low energy ($k/\beta = 0.25$) and the lower set to high energy ($k/\beta = 2.0$) scattering respectively. (For the latter case, we have divided the cross sections by 10 and have only shown the curves for three values of α for clarity.) The lower energy curves show more structure, essentially brought about by the interference term. From figure 1, it is clear that for $\alpha = 0$ and $\alpha = 1$, the cross section is symmetric about $\theta = \pi/2$. In two dimensions, parity transformation may be taken to correspond to $x \to -x$ and $y \to y$, or $\theta \to (\pi - \theta)$. Thus asymmetry in the cross section about $\theta = \pi/2$ signals parity violation. This violation is brought about by the interference term, and the asymmetry is clearly seen in figure 1. The other point that clearly comes out from the figure is that the fermionic cross section goes to zero for $\theta = \pi/2$, which is true for spinless fermions.

Most interesting is the appearance of a pronounced dip in the forward angle region for low-energy scattering of two anyons, as seen in the upper set of curves in the figure. In our problem, since anyons are taken to be identical charged particles, $\beta = me^2/\hbar^2$ is always positive, and the planar Coulomb part of the interaction is repulsive. However, changing the sign of β mathematically results in the interference dip shifting to $\theta > \pi/2$ region†. Note that for bosons ($\alpha = 0$), there are clear symmetrical dips in the cross

† It is of some mathematical interest to note that for an attractive planar Coulomb potential between anyons, the two-body bound-state problem is easily solved, and exhibits level-crossing at $\alpha = \frac{1}{2}$.

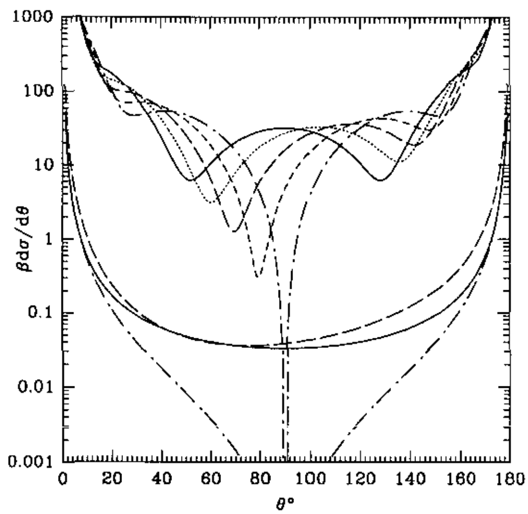

Figure 1. The Mott anyon–anyon differential cross sections ($\beta\, d\sigma/d\theta$) are displayed versus scattering angle $\theta°$, for $\alpha = 0$ (——), $\alpha = \frac{1}{4}$ (· · ·), $\alpha = \frac{1}{2}$ (— — —), $\alpha = \frac{3}{4}$ (- - - -), and $\alpha = 1$ (— - - —). The upper set of curves corresponds to the low-energy scattering with $k/\beta = 0.25$ and the lower set of curves corresponds to high-energy scattering with $k/\beta = 2.0$, and we have displayed only the $\alpha = 0, \frac{1}{2}, 1$ curves, and divided the cross sections by 10 for the latter set.

section on both sides of $\theta = \pi/2$. As α is increased, the forward angle dip shifts more and more towards $\theta = \pi/2$, approaching the null cross section for $\alpha = 1$. This effect is spectacular for low-energy scattering, and is of statistical origin. Such extreme asymmetry in scattering, together with the pronounced dip may be taken to be the signature of anyons, but it is not clear if it is at all possible to exploit this experimentally.

This research was supported by a grant from the Natural Sciences and Engineering Research Council (Canada). Thanks are due to Dr M V N Murthy for very helpful discussions. JL and MKS would also like to acknowledge the hospitality of the Physics Department, McMaster University.

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